Reputation entrenchment or risk minimization?

Early stop and investor-manager agency conflict in fund management

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Abstract
Purpose – One of the agency conflicts between investors and managers in fund management is reflected by risk-taking behaviors led by their different goals. The investors may stop their investments in risky assets before the end of the investment horizon to minimize risk, while the managers may do so to entrench their reputation so as to pursue better opportunities in the labor market. This study aims to consider a one principal-one agent model to investigate this agency conflict.

Design/methodology/approach – The paper derives optimal asset allocation strategies for both parties by extending the traditional dynamic mean-variance model and considering possibilities of optimal early stopping. Doing so illustrates the principal-agent conflict regarding risk-taking behaviors and managerial investment myopia in fund management.

Practical implications – This paper not only paves the way for further studies along this line, but also presents results useful for practitioners in the money management industry.

Findings – According to the theoretical analysis and numerical simulations, the paper shows that potential early stop can make the agency conflict worsen, and it proposes a way to mitigate this agency problem.

Originality/value – As one of the exploratory studies in investigating agency conflict regarding risk-taking behaviors in the literature, this study makes multiple contributions to the literature on fund management, asset allocation, portfolio optimization, and risk management.

Keywords Fund management, Investments, Risk management, Modelling

Paper type Research paper

1. Introduction
In the last three decades, rapid growth of funds, in both numbers and size of assets, has been realized by both academic researchers in financial economics and practitioners in financial markets. Issues in fund management, such as risk, performance, and managerial incentive-related ones, have been examined by prior research (e.g. Brown *et al.*, 2001; Goetzmann *et al.*, 2003). However, the investor-manager agency conflict regarding their risk-taking behaviors in the field of fund management has just started attracting the attention of financial economists. Basak, Shapiro and Tepla (2006) and Basak, Pavlova and Shapiro (2006a, b) are pioneer studies on this issue, and they focus

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on risk management with benchmarking, risk-shifting, optimal asset allocation, and managerial incentives in money management.

Manager's investment myopia, which induces her not to act in the interest of investors, is one of the major agency problems in this field since it is closely related to her career concerns. Doing so may help entrench her reputation as early as possible and give her chances to pursue better opportunities in the labor market. This incentive generates a possibility of early stop in her investment strategy and risk-taking behaviors. On the other hand, investors may also want to exit the stock markets early as long as their expected returns at the end can be guaranteed because doing so may help them minimize risk. Thus, the investor-manager agency conflict can be caused by their different objectives of early optimal stopping. Additionally, risk measurement in the fund management industry is also of interest because of its dynamic nature. Lo (2001) claims that none of the static risk measurements fits the analysis on fund management, and therefore a dynamic measure needs to be used to properly describe its nature.

The mean-variance approach first proposed by Markowitz (1952, 1959) is the cornerstone of modern finance theory, and is the foundation of some major areas in finance such as portfolio selection, capital market and risk management. This two-dimensional model abstracts the trade-off between the expected return and the corresponding risk. Markowitz's (1952, 1959) Noble-winning studies present this critical trade-off in a single-period setting by minimizing risk measured by variance given an expected return, and his model has been extended to multi-period (e.g. Hakansson, 1971; Samuelson, 1986) and dynamic settings (e.g. Elliott and Kopp, 1999; Li and Ng, 2000; Li, 2000). Zhou and Li (2000) is one of the classic studies addressing continuous-time mean-variance portfolio selection, and this issue is further discussed by Bielecki *et al.* (2005) by taking bankruptcy prohibition into account. Some recent work (Pástor, 2000; Sundaresan, 2000; Li *et al.*, 2001) summarizes the development of portfolio optimization and the continuous-time modeling literature, and re-highlights the importance of the mean-variance model in finance.

In the prior research on optimal asset allocation and portfolio selection, financial economists usually assume that an investor does not stop investing in risky assets until the terminal point for analytical simplicity, and determines her optimal investment strategies by minimizing risk for a given level of expected return at the terminal time. Although different assumptions and sophisticated constraints, such as time-dependent investment strategies, alternative measures of risk, short-selling and bankruptcy constraints, and so on, have been imposed to the expected return-risk framework, an investor is usually assumed to be active in financial markets all the time during the investment horizon. In practice, however, this is not always true as an investor can choose to stop investing in risky assets at an optimal time before the end of investment horizon and put all her money into risk-free assets. This is similar to the case in which a person who thinks herself wealthy enough may decide to retire early. To our best knowledge, this potential has not been extensively addressed in the portfolio selection literature, although it is of interest.

In this study, we extend the well-known dynamic mean-variance model to a new stage for deriving optimal asset allocation strategies by considering a possible early stop depending on different optimal stopping criteria of investors and managers. The logic for an investor to stop early is based on risk minimization given her expected level of terminal return, while that for a manager is reputation entrenchment. Thus, we present a new economic setting to hedge risk more effectively in portfolio optimization for the investor by illustrating the effects of an optimal stopping on asset allocation strategies, and in the meantime, we discover the investor-manager agency conflict regarding their risk-taking behaviors worsened by two parties' different incentives for early stop. Therefore, we study the agency problems in fund management in a different view from those in Basak *et al.* (2006) and Basak, Pavlova and Shapiro (2006a, 2006b).

Considering this interesting optimal stopping problem in asset allocation, we refer its nature to American-option-like, if we view traditional portfolio selection problems as European-option-like ones. To derive optimal asset allocation strategies for both parties considering a possibility of early stop, we present a variance-minimization framework in an Arrow-Debreu system and convert it to an American-option-like problem. Because of the existence of a potential optimal stopping, which is similar to the early exercise in American option valuation, however, deriving an optimal asset allocation strategy is mathematically much more complicated than the traditional problems in the field. Thus, techniques for pricing American-style securities need to be adopted and only approximate solutions can be found. Using the mathematical tools such as martingale, we derive an approximate solution to the American-style problem, and highlight the nature of optimal stopping by comparing the variances of terminal wealth under different situations.

Multiple contributions are made to the literature by the current study. First, it adds to the fund management literature by combining the agency problems and dynamic risk measurement. It is of interest for both financial economists and practitioners in the fund management industry, and of importance to better understand and to mitigate the investor-manager agency conflict. Second, it contributes to the investment literature by restructuring the traditional asset allocation problem setting and addressing a possibility of optimal stopping. As an exploratory study, it does not only push the academic research closer to the real world, but also paves the road for further research in this field using alternative risk measurements and/or under different constraints. Third, it sheds some light on the risk-hedging and risk-management literature by claiming that risk taken by an investor can be more effectively reduced in the presence of optimal stopping, and therefore it makes her better off. Fourth, it combines the American-option pricing and the asset allocation literature to enrich the implications of finance theory, and applies techniques of pricing American-style securities to the portfolio optimization literature. It also extends the American option literature to a broader context, and makes the techniques for valuing American-style securities more significant.

This article is organized as follows: section 2 discusses the investor-manager agency conflict in fund management, and describes the economic setting for the asset allocation problem with potential early stop in a dynamic mean-variance framework. To highlight the nature of optimal stopping in the above framework, we first derive it in a traditional mean-variance model without considering early stop in section 3.1, followed by sections 3.2 and 3.3 which present approximate solutions to the optimal asset allocation problems with optimal stopping for both investors and managers, respectively. Section 4 illustrates the nature of the investor-manager agency conflict worsened by the different objectives of investors and managers for early stop by comparing their risk-taking behaviors. Section 5 provides concluding remarks and recommendations for further research.

2. The economic setting

2.1. Investor-manager agency conflict in fund management

In fund management, investor-manager conflict has been realized in the literature, and performance-related incentive compensation is widely used to mitigate it. However, agency problems in this field have not been completely addressed because previous studies focus more on the return side in the risk-return trade-off. One has realized that this agency conflict can also be reflected by different risk-taking behaviors, and therefore, this study attempts to discover this problem in an agency framework, and discusses how risk-sharing can mitigate this problem.

In a one investor-one manager framework, the investor tries to minimize her risk taken for implementing investment strategies so as to guarantee a pre-fixed level of expected terminal return over an investment horizon. Thus, as long as her wealth is higher than or equal to her expected terminal wealth, discounted by the risk-free rate at one point of time before the terminal point, she may exit the stock markets and invest all of her wealth into risk-free assets. Doing so cannot only guarantee her expected terminal return at the end of the investment horizon, but also help her reduce risk. We refer this point to an optimal stopping time for the investor. Note that if there happens to have no such a point before the end, the investor will not stop investing in risky assets until the terminal point.

However, the manager does not necessarily act in the investor's interest, but tries to entrench her reputation as soon as possible. She may want to stop early as well so that she can renegotiate with the investor and/or pursue a better opportunity in the labor market. To do so, the manager's optimal early stop criterion is not necessarily the same as that for the investor's early stop, while to stop early, the manager has to take higher risk in her investment so as to outperform other competitors in the market.

Thus, there is an agency conflict between the investor and the manager regarding their risk-taking behaviors caused by their different objectives. To further discover this agency problem, we extend the traditional dynamic mean-variance model to a new stage by considering possibilities of optimal early stops before the terminal point for both parties. Since the investor attempts to minimize her risk but the manager tries to entrench her reputation as soon as possible, they use different criteria to determine the optimal time at which they stop investing in risky assets.

2.2. A dynamic mean-variance economic setting

To be one of the exploratory studies regarding the investor-manager agency conflict in fund management generated by the possibility of optimal early stopping, we attempt to make the problem setting as simple as possible, as long as it can illustrate the nature of the problem. Following Zhou and Li (2000) and Li *et al.* (2001), we consider a financial market wherein $m+1$ assets, a risk-free bond and m risky stocks, can be traded continuously over a finite horizon $[0,T]$. We also assume that there is no transaction cost, no income tax, no asymmetric information in the market, and there is no restriction on borrowing and lending at the risk-free rate by investors. Furthermore, an investor with initial wealth X_0 is a price taker, and therefore her actions do not affect the probability distributions of security returns. Perfect short selling is allowed, and bankruptcy is allowed in the view of investors, which means that at any point of time t the investor's wealth can be negative. From the manager's point of view, however, the fund assets at any point of time t should be nonnegative (i.e. $X(t) \geq 0$) because otherwise she will be fired and get a bad reputation in the labor market.

To clarify the notations used in the rest of the article and to make them consistent with previous studies in the literature (e.g. Zhou and Li, 2000; Li *et al.*, 2001), we define the economic setting using the following notations: M' is the transpose of any matrix or vector M, and |M| is $\sqrt{\sum_{i,j} m_{i,j}^2}$ for any matrix or vector $M = (m_{ij})$. x_+ is max(x,0) for any real number x, while x_- is max($-x$,0). $\mathbb{R}^n$ is an n-dimensional real Euclidean space.

The risk-free bond price at time t is denoted by $S_0(t)$, $t \geq 0$, and its dynamics are governed by the ordinary differential equation:

$$\begin{cases} dS_0(t) = r(t)S_0(t)dt, \\ S_0(0) = s_0, \end{cases}$$

where $r(t) \geq 0$ is its interest rate at time t. The m non-dividend-paying stock prices over the investment horizon [0, T] are assumed to be log-normally distributed and follow standard Brownian motions, and they are modeled by the stochastic differential equations:

$$\begin{cases} dS_i(t) = S_i(t)\{\mu_i(t)dt + \nu_i(t)dz_i(t)\}, \ i = 1, \cdots, m, \\ S_i(0) = s_i, \end{cases} \tag{2.1}$$

where $\mu(t) := (\mu_1(t), \cdots, \mu_m(t))'$ is the drift rate vector, $\nu(t) := \text{diag}(\nu_1(t), \cdots \nu_m(t))$ is the volatility matrix, and $z(t) := (z_1(t), \cdots, z_m(t))'$ is an m-dimensional Brownian motion with the correlation coefficient matrix $\rho(t)$. To simplify the derivation of the optimal asset allocation strategy, one can rewrite equation (2.1) as:

$$\begin{cases} dS_i(t) = S_i(t)\left\{\mu_i(t)dt \sum_{j=1}^{m} \sigma_{ij}(t)dW^j(t)\right\}, \ t \in [0, T], \ i = 1, \cdots, m, \\ S_i(0) = s_i, \end{cases} \tag{2.2}$$

where $W(t) : -(W_1(t), \cdots, W_m(t))'$ is an m-dimensional standard Brownian motion. The coefficient matrix $\sigma(t) := (\sigma_{\text{ij}}(t))$ in (2.2) satisfies the non-degeneracy condition:

$$\sigma(t)\sigma(t)' = \nu(t)'\rho(t)\nu(t) \geq \delta I, \ \forall t \in [0, T],$$

where $\delta > 0$ is a given constant. Following Duffie and Richardson (1991), we assume that, for analytical simplicity, $r(t)$, $\mu(t)$, $\nu(t)$ and $\rho(t)$ are deterministic, Borel-measurable, and bounded on the investment horizon [0, T]. The above setting for security prices is what has been widely adopted by previous studies in the literature on continuous-time mean-variance portfolio selection (e.g. Zhou and Li, 2000; Li *et al.*, 2001; Bielecki *et al.*, 2005; Li and Zhou, 2006).An investor, whose initial wealth is $X_0 > 0$, is risk averse, and she makes her investment decision on the basis of the terminal value of her portfolio. Her dynamic optimization problem is to minimize risk given a level of expected return, and the solution to it is an optimal asset allocation strategy for constructing a portfolio by allocating her wealth among the $m+1$ assets with different levels of risk. We denote a trading strategy for the investor as an m-dimensional process $\{\pi(t):0 \leq t \leq T\}$ whose ith component, $\pi i(t)$, is the value of the holdings of risky asset i in the asset portfolio at time t. As shown in previous studies (e.g. Karatzas and Shreve, 1998; Elliott and Kopp, 1999; and Zhou and Li, 2000), under an admissible trading strategy $\pi(t)$, the value of the fund assets $X(t)$, which is the investor's wealth at time t, follows:

$$\begin{cases} dX(t) = \left\{ \left[r(t)\left(X(t) - \sum_{i=1}^{m} \pi_i(t)\right) + \sum_{i=1}^{m} \mu_i(t)\pi_i(t) \right] dt + \sum_{j=1}^{m}\sum_{i=1}^{m} \sigma_{ij}(t)\pi_i(t) dW^j(t) \right\} \\ X(0) = X_0. \end{cases}, \quad (2.3)$$

This implicitly assumes that the investor does not consume any of her wealth over the investment horizon.

Note that, with perfect short selling, neither $\pi 0(t)$ nor $\pi_i(t)$ has to be non-negative, and a negative $\pi_0(t)$ means that the investor borrows money at the risk-free rate to invest in risky assets. For the sake of brevity, we follow Zhou and Li (2000) and Li *et al.* (2001) to rewrite the system (2.3) in vector form as:

$$dX(t) = \left(r(t)X(t) + \pi(t)' B(t) \right) dt + \pi(t)' \sigma(t) dW(t), \quad (2.4)$$

where $B(t) = \mu(t) - r(t)\mathbf{1}$,, and to present the dynamic mean-variance portfolio selection problems of the investor and the manager. $\mathbf{1}$ is the m-dimensional column vector with each component equal to 1.

The investor's optimization problem in the dynamic mean-variance economy parameterized by $z \geq X_0 e^{\int_0^T r(s)ds}$ is presented as:

$$\min_{0 \leq t \leq T} \mathrm{Var}\left[X(t) e^{\int_t^T r(s)ds} \right], \text{subject to} \begin{cases} \mathrm{E}\left[\mathrm{X(t)} e^{\int_t^T r(ds)} \right] = z, \\ \pi(\cdot) \in L^2_{\mathcal{F}}\left(0, T; \mathbb{R}^m\right), \\ \left(X(\cdot), \pi(\cdot)\right) \text{ satisfy (2.4)}, \end{cases} \quad (2.5)$$

and that of the manager is:

$$\min_{0 \leq t \leq T} \mathrm{Var}\left[X(t) e^{\int_t^T r(s)ds} \right], \text{subject to} \begin{cases} \mathrm{E}\left[\mathrm{X(t)} e^{\int_t^T r(ds)} \right] = z, \\ X(t) \geq 0 \text{ a.s.}, \\ \pi(\cdot) \in L^2_{\mathcal{F}}\left(0, T; \mathbb{R}^m\right) \\ \left(X(\cdot), \pi(\cdot)\right) \text{ satisfy (2.4)}, \end{cases} \quad (2.6)$$

If we denote τ as the optimal stopping time, τ_I according to the investor's criterion and τ_M according to the manager's, the optimal asset allocation strategy of (2.5) (and (2.6), respectively) is called an efficient strategy, and $(\mathrm{Var}[X^*(\tau)e^{\int_\tau^T r(s)ds}], z)$, where $\mathrm{Var}[X^*(\tau)e^{\int_\tau^T r(s)ds}]$ is the optimal value of (2.5) (and (2.6), respectively) corresponding to z and τ, is called an efficient point. The set of all efficient points forms the efficient frontier.

2.3 Applications of the martingale method in optimization problems

The dynamic mean-variance portfolio problem defined above (2.5) (and (2.6), respectively) has been solved using approximation methods to simulate optimal investment strategies. Following Cox and Huang (1989), we adopt the martingale method to derive optimal asset allocation strategies and present in Section 3.1 exact solutions. It is a preparation for illustrating the effects of optimal stopping on the investor-manager agency conflict by comparing them to the optimal strategies presented in sections 3.2 and 3.3 in the presence of early stop.

Denote $\phi(t)$ as a state price density in an Arrow-Debreu system. The martingale method requires:

$$E\left[\phi(s)X(s)|\mathcal{F}_t|\right] = \phi(t)X(t),\ s > t, \tag{2.7}$$

where:

$$\begin{cases} d\phi(t) = \phi(t)\{-r(t)dt - \theta(t)' dW(t)\}, \\ \phi(0) = 1 \end{cases} \tag{2.8}$$

and the relative risk premium process defined by $\theta(t) \equiv \sigma(t)^{-1}B(t)$ when $E(e^{\frac{1}{2}\int_0^T |\theta(t)|^2 dt}) < \infty$ is sufficed by Novikov's condition. According to Harrison and Kreps (1979), the no-arbitrage constraint and the complete market assumption guarantee the existence and the uniqueness of $\phi(t)$. The density function of process $\phi(\cdot)$ is:

$$p\left(\phi(T)\right) = \frac{1}{\phi(T)\sqrt{2\pi\int^T |\theta(s)|^2 ds}} \exp\left(-\frac{\left(\ln\phi(T) - \ln\phi(t) + \int_t^T \left(r(s) + 1/2|\theta(s)|^2\right)ds\right)^2}{2\int_t^T |\theta(s)|^2 ds}\right). \tag{2.9}$$

Using the state price density $\phi(t)$, we find that the dynamic budget constraints of (2.4) are equivalent to the static budget constraint, $E[\phi(t)X(t)] = X_0$, and therefore, the dynamic optimization problem can be decomposed into two stages with a static optimization problem in the first stage. The optimal wealth x^* according to early stop, which is the value of the wealth $X(\tau)$ determined by optimal portfolio at the optimal stopping time τ, is derived from the first-stage static optimization problem for the investor with a random stopping time $\tau_I \in [0, \mathrm{T}]$:

$$\min \mathrm{Var}\left[X(\tau_I)e^{\int_{\tau_I}^T r(s)ds}\right], \text{subject to} \begin{cases} E\left[X(\tau_I)e^{\int_{\tau_I}^T r(s)ds}\right] = z, \\ E\left[\phi(\tau_I)X(\tau_I)\right] = X_0. \end{cases} \tag{2.10}$$

Then the corresponding strategy $\pi_I(\cdot)$ that replicates $X_I(\cdot)$ can be found via the dynamic equation:

$$\begin{cases} dX(t) = \left[r(t)X(t) + \pi(t)'B(t)dt + \pi(t)'\sigma(t)dW(t)\right\}, \\ X(\tau_I) = x_I^*, \end{cases} \quad (2.11)$$

where x_I^* is the optimal value of wealth in the investor's optimization problem (2.10). Note that the above (2.10) and (2.11) present general cases with an optimal stopping before or at the terminal point for the investor. When the investment in risky investment is not stopped until the terminal time (i.e. $\tau_I = T$), the optimization problem is a European-style, and the solution to it is presented in Section 3.1. Otherwise, the existence of early stop affects the investment strategy and risk management from the investor's point of view, and it is an American-style[1] problem whose solution is presented in Section 3.2.

Similarly, the optimization problem for the manager with a random optimal stopping time $\tau M \in [0, T]$ is:

$$\min \mathrm{Var}\left[X(\tau_M)e^{\int_{\tau_M}^{T} r(s)ds}\right], \text{subject to} \begin{cases} \mathrm{E}\left[\mathrm{X}(\tau_\mathrm{M})e^{\int_{\tau_M}^{T} r(s)ds}\right] = z, \\ E\left[\phi(\tau_M)X(\tau_M)\right] = X_{0,} \\ X(\tau_M) \geq 0. \end{cases} \quad (2.12)$$

Then the corresponding strategy $\pi_M(\cdot)$ that replicates $X_M(\cdot)$ can be found via the dynamic equation:

$$\begin{cases} dX(t) = \left[r(t)X(t) + \pi(t)'B(t)dt + \pi(t)'\sigma(t)dW(t)\right\}, \\ X(\tau_M) = x_M^*, \end{cases} \quad (2.13)$$

where x_M^* is the optimal value of portfolio in the manager's optimization problem (2.12). Similar to the above case for the investor, (2.12) and (2.13) present general cases with an optimal stopping before or at the terminal point for the manager. When the investment in risky assets is not stopped until the terminal time (i.e. $\tau_M = T$), the optimization problem (2.12) has a European style, and the solution to it is presented in section 3.1. Otherwise, the existence of optimal stopping affects the investment strategy and risk management in the view of manager, and it is an American-style problem whose solution is presented in section 3.3.

3. Optimal asset allocation strategies

According to the dynamic mean-variance economic setting described in section 2.2, one realizes that the existence of an early stop is potentially influential on the optimal asset allocation strategy chosen by an investor or by a manager. In this section, we first derive the exact solutions to the mean-variance optimization problems (2.10) and (2.12), respectively, without considering the possibility of early stop, and then present the approximate solutions to them in the presence of optimal stopping.

3.1. Optimization problems without early stop

As discussed previously, we first solve for the optimal terminal wealth $X_i^*(\cdot)$ which satisfies the backward stochastic differential equation (BSDE) (2.11) for the investor in this two-staged optimization method without considering a possibility of early stop, and then find the investor's optimal asset allocation strategy $\pi_i^*(\cdot)$ along with the wealth process $X_i^*(\cdot)$ via the second-stage problem. The existence and uniqueness of an admissible strategy satisfying (2.11) are, as stated above, guaranteed by the BSDE theory. Note that the asset allocation strategies presented in this subsection are similar to those presented and discussed in previous studies (e.g. Zhou and Li, 2000; Li and Zhou, 2006), while the approach adopted in this study is different from those presented in them.

Theorem 1 below presents the solution to the investor's risk-minimization problem without allowing an early stop, including both $X_i^*(\cdot)$ and $\pi_i^*(\cdot)$. See Appendix 1 for the detailed proof.

Theorem 1. Assume that $\int_0^T|\theta(s)^2 ds > 0$. Then the optimal terminal wealth (2.10) without early stop is presented by:

$$x_I + \lambda_I \phi(T), \tag{3.1}$$

where:

$$\begin{cases} \lambda_I = \frac{zE\left[\phi(T)^2\right] - X_0 E\left[\phi(T)\right]}{\mathrm{Var}\left[\phi(T)\right]}, \\ \gamma_I = \frac{zE\left[\phi(T)\right] - X_0}{\mathrm{Var}\left[\phi(T)\right]}. \end{cases} \tag{3.2}$$

Given the Lagrange multipliers λ_I, γ_I above, the investor has a unique efficient portfolio for (2.5) corresponding to her optimal wealth (3.1) without a possibility of early stop. Moreover, the efficient portfolio and associated wealth process are given respectively by:

$$\pi_I(t) = \left(\sigma(t)\sigma(t)'\right)^{-1} B(t) S_I(t) \tag{3.3}$$

and

$$X_I(t) = \lambda_I e^{-\int_t^T r(s)ds} - S_I(t), \tag{3.4}$$

where $S_I(t) = \gamma_I \phi(t) e^{\int_t^T (-2r(s) + |\theta(s)|^2) ds}$.

Thus, theorem 1 presents the investor's optimal investment strategy in the dynamic mean-variance economic setting defined in section 2.2 while early stop is not allowed. It describes the investor's optimal asset allocation strategy among the risk-free asset and m risky assets at any time t over the investment horizon $[0, T]$. This is not, however, the same as that for the manager under the same situation since the manager cannot afford on bankruptcy. If the fund asset managed by her was negative at any point of time t, she would be fired and her reputation would be damaged.

To tell the difference between their risk-taking behaviors, we use the same two-staged optimization method to solve for the optimal wealth process $X_M^*(\cdot)$ and the optimal investment strategy $\pi_M^*(\cdot)$ along with it for the manager without considering a possible early stop. The wealth process $X^*_M(\cdot)$ follows the BSDE (2.13). Theorem 2 below presents the solution to the manager's optimization problem without allowing early stop, including both $X^*_M(\cdot)$ and $\pi^*_M(\cdot)$. See Appendix 2 for the detailed proof.

Theorem 2. Assume that $\int_0^{\tau_M} |\theta(s)|^2 ds > 0$. Then the value of optimal fund assets (2.12) is:

$$x_{\tau_M} = e^{\int_{\tau_M}^{T} -r(s)ds}\left(\lambda_{\tau_M} - \gamma_{\tau_M}\phi(\tau_M)e^{\int_{\tau_M}^{T} -r(s)ds}\right)+, \tag{3.5}$$

where $(\lambda_{\tau M}, \gamma_{\tau M})$ is the unique solution to the following system of equations:

$$\begin{cases} \lambda_{\tau_M} N\left(\dfrac{\ln\left(\lambda_{\tau_M}/\gamma_{\tau_M}\right)+\int_0^T r(s)ds+\int_0^{\tau_M} 1/2|\theta(s)|^2 ds}{\sqrt{\int_0^{\tau_M}|\theta(s)^2 ds|}}\right) \\ -\gamma_{\tau_M} e^{\int_0^T -r(s)ds} N\left(\dfrac{\ln\left(\lambda_{\tau_M}/\gamma_{\tau_M}\right)+\int_0^T r(s)ds-\int_0^{\tau_M} 1/2|\theta(s)|^2 ds}{\sqrt{\int_0^{\tau_M}|\theta(s)^2 ds|}}\right) \\ \lambda_{\tau_M} e^{\int_0^T -r(s)ds} N\left(\dfrac{\ln\left(\lambda_{\tau_M}/\gamma_{\tau_M}\right)+\int_0^T r(s)ds-\int_0^{\tau_M} 1/2|\theta(s)|^2 ds}{\sqrt{\int_0^{\tau_M}|\theta(s)^2 ds|}}\right) \\ -\gamma_{\tau_M}\phi(t)e^{\int_0^T -2r(s)ds+\int_0^{\tau_M}|\theta(s)|^2 ds} N\left(\dfrac{\ln\left(\lambda_{\tau_M}/\gamma_{\tau_M}\right)+\int_0^T r(s)ds-\int_0^{\tau_M} 3/2|\theta(s)|^2 ds}{\sqrt{\int_0^{\tau_M}|\theta(s)^2 ds|}}\right) = X_0. \end{cases} \tag{3.6}$$

Given the Lagrange multipliers $(\lambda_{\tau_M}, \gamma_{\tau_M})$ above, the manager has a unique efficient portfolio for (2.6) corresponding to her optimal wealth process (3.5) without early stop. Moreover, the efficient portfolio and associated fund asset process are given respectively by:

$$\pi_{\tau_M}(t) = \left(\sigma(t)\sigma(t)'\right)^{-1} B(t)N\left(-d_1^{\tau_M}\left(t, S_{\tau_M}(t)\right)\right)S_{\tau_M}(t), \text{ for } 0 \le t \le \tau_M \tag{3.7}$$

and

$$X_{\tau_M}(t) = \lambda_{\tau_M} e^{-\int_t^{\tau_M} r(s)ds} N\left(-d_2^{\tau_M}\left(t, S_{\tau_M}(t)\right)\right) - S_{\tau_M}(t)N\left(-d_1^{\tau_M}\left(t, S_{\tau_M}(t)\right)\right), \tag{3.8}$$

where $N(\cdot)$ is the cumulative distribution function of the standard normal distribution and:

$$\begin{cases} N(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{u^2/2} du, \\ S_{\tau_M}(t) := \gamma_{\tau_M} \phi(t) e^{\int_t^T -2r(s)ds + \int_t^{\tau_M} |\theta(s)|^2 ds}, \\ \phi(t) := e^{\int_0^t -r(s) - 1/2 - |\theta(s)|^2 - \theta(s)' dW(s) ds}, \\ d_1^{\tau_M}(t, S_{\tau_M}) := \frac{\ln S_{\tau_M}(t) - \lambda_{\tau_M} + \int_t^T r(s)ds + \int_t^{\tau_M} 1/2 |\theta(s)|^2 ds}{\sqrt{\int_t^{\tau_M} |\theta(s)|^2 ds}}, \\ d_2^{\tau_M}(t, S_{\tau_M}) := d_1^{\tau_M}(t, S_{\tau_M}) - \sqrt{\int_t^{\tau_M} |\theta(s)|^2 ds}. \end{cases} \quad (3.9)$$

In the next two sub-sections, we discuss the influence of early stop on optimal investment strategies for the investor and the manager, respectively, according to their own optimal stopping criteria. As stated in the introduction, we can consider this interesting optimal stopping problem in the asset allocation as an American-option-like security, if we view that the nature of traditional portfolio selection problems is European-option-like. Because of the existence of a potential early stop, which is similar to early exercise in an American option, deriving an optimal asset allocation strategy is mathematically much more complicated than the traditional problems in the field, and therefore, techniques for pricing American-style securities need to be adopted.

The option valuation literature is developed from Black and Scholes (1973) seminal work which provides closed-form expressions of European options in an arbitrage framework by assuming a log-normal distribution of underlying stock returns, while the risk-neutral valuation approach introduced by Cox and Ross (1976) and extended by Harrison and Kreps (1979) significantly affect the subsequent studies in this area.

Unfortunately, the optimal stopping problem due to their special feature of the early exercise potential prevents financial economists from valuing American-style derivatives analytically, unless some strict assumptions, such as zero or discrete predictable dividend payments, perpetual investment horizon, or some specific processes (e.g. the Lévy process and the fractional Brownian motion), are made (e.g. Boyarchenko and Levendorskii, 2002; Elliott and Chan, 2004). In the absence of a generalized closed-form expression valuing an American option with finite maturity and continuous dividend payments, various simulation-based numerical approaches for solving American option valuation problems. They can mainly be categorized into the binomial tree approach rooting in Cox *et al.* (1979), the finite difference approximations pioneered by Brennan and Schwartz (1977) and Schwartz (1977), the quasi-analytical approaches introduced by Geske and Johnson (1984), and Monte-Carlo simulations based on Boyle (1977).

In the next two sub-sections, we present approximate solutions to the mean-variance optimization problems with optimal stopping for both the investor and the manager.

3.2. Investor's optimization problem with early stop

One realizes that in the economy described above, an investor can exit the stock market at any time as long as she can get at least her expected return in the future by investing all of her wealth into risk-free assets. Remind that she can do so due to the no-consumption assumption we have made above without loss of generality. Since stock prices move randomly, she can compare her instantaneous wealth at any point of time with her expectation, and decide whether she should stay in the stock markets or not.

We define the investor's optimal wealth without optimal stopping as $X_I(t) = f_I(t, T; S_I(t))$, where $f_I(t, T; S_I(t)) = \lambda e^{\int_t^T r(s)ds} - S_I(t)$. To examine the case with a possibility of optimal stopping before the terminal point, we need to construct a wealth process to $X_I^*(\cdot)$ satisfying:

$$X_I^*(t) = \begin{cases} X_I(t), & \text{if } 0 \le t \le \tau_I \\ e^{\int_{\tau_I}^t r(s)ds} X(\tau_I), & \text{if } \tau_I < t \le T, \end{cases}$$

where τ_I is the investor's optimal stopping time as defined above.

As long as the future value of her expected wealth at time T can be reached by investing her wealth into the risk-free asset from time τ_I, the investor immediately exits the stock markets and does not take any additional risk after that point of time. Remind that the investor's optimal wealth in the non-early stop situation, $X(\cdot)$, presented in theorem 1 satisfies the following problem:

$$\tau_I := \inf\left\{ t \ge 0 \middle| \lambda e^{\int_t^T r(s)ds} - S_I(t) = z e^{-\int_t^T r(s)ds} \right\} \wedge T. \tag{3.10}$$

Based on this criterion, we derive the optimal investor's optimal asset allocation strategy with a potential early stop as presented in the theorem below.

Theorem 3. Assume that $\int_0^T |\theta(s)|^2 ds > 0$ and that the Lagrange multipliers λ_I, γ_I are given by (3.2). Then there is an approximate efficient portfolio for (2.5) corresponding to an early stop time τ_I. Moreover, the efficient portfolio and associated wealth process are given respectively by:

$$\pi_I^*(t) \begin{cases} \left(\sigma(t)\sigma(t)'\right)^{-1} B(t) S_I(t), & \text{if } 0 < t \le \tau_I, \\ 0, & \text{if } \tau_I < t \le T \end{cases}$$

and:

$$X_I^*(t) \begin{cases} \lambda e^{-\int_t^T r(s)ds} - S_I(t), & \text{if } 0 < t \le \tau_I, \\ \lambda e^{\int_t^T r(s)ds} - S_I(\tau) e^{\int_{\tau_I}^t r(s)ds} & \text{if } \tau_I < t \le T. \end{cases}$$

From theorem 1 and theorem 3, one immediately sees the difference between the two optimal asset allocation strategies for the investor under different assumptions regarding the existence of a potential optimal stopping. If the investor stops investing in risky assets before time T and stays in the risk-free bond market only, furthermore, she does not take any additional risk after that with guaranteeing the pre-determined expected return at the terminal time T. Therefore, her risk measured by variance should be lower than that in the traditional non-early stop situation. This is shown in section 4.1 by comparing the variances in different cases numerically without loss of generality.

3.3. Manager's optimization problem with early stop

Similar to the investor's behaviors described above, the manager can also choose to exit the stock markets at any time as long as her optimal stopping criterion is met. We define the manager's optimal fund asset without early stop as $X^*_{T_M}(t) = f_{T_M}(t, T; S_{T_M}(t))$, where:

$$f_{T_M}\left(t, T; S_{T_M}(t)\right) = \lambda_T e^{-\int^T r(s)ds} N\left(-d_2^{T_M}\left(t, S_{T_M}(t)\right)\right)$$
$$- S_{T_M}(t) N\left(-d_1^{T_M}\left(t, S(t)\right)\right).$$

To examine the case with the possibility of optimal stopping before the maturity T, we need to construct a wealth process to $X_{\tau_M}(t) = f_{\tau_M}(t, \tau_M; S_{\tau_M}(t))$, which is a new option price according to possible maturity time $0 \leq \tau$le; T.

This wealth process $X_{\tau_M}(\cdot)$ satisfies:

$$\begin{cases} X_{\tau_M(t)} = \begin{cases} f_{\tau_M}\left(t, \tau_M; S_{\tau_M}(t)\right), & \text{if } 0 \leq t \leq \tau_M, \\ e^{\int_{\tau_M}^{t} r(s)ds} f_{\tau_M}\left(\tau_M, \tau_M; S_{\tau_M}(\tau_M)\right), & \text{if } \tau_M < t \leq T, \end{cases} \\ E\left|X_{\tau_M}(T)\right| = z, \end{cases}$$

where:

$$f_{\tau_M}\left(t, \tau_M; S_\tau(t)\right) = \lambda_{\tau_M} e^{-\int_t^{\tau_M} r(s)ds} N\left(-d_2^{\tau_M}\left(t, S_{\tau_M}(t)\right)\right) - S_{\tau_M}(t) N\left(-d_1^{\tau_M}\left(t, S_{\tau_M}(t)\right)\right),$$

for $0 \leq t \leq \tau_M$, and $f_{\tau_M}(\tau_M, \tau_M; S_{\tau_M}(\tau_M))$ indicates the payoff to the investor at time τ_M, i.e.:

$$f_{\tau_M}\left(\tau_M, \tau_M; S_{\tau_M}(\tau_M)\right) = \left(\lambda_{\tau_M} - S\tau_M(\tau_M)\right) + .$$

Thus, as long as the manager's optimal stopping criterion:

$$\hat{\tau}_M := \inf\left\{\tau_M \geq 0 \middle| f_T\left(\tau_M, T; S_T(\tau_M)\right) \leq f_{\tau_M}\left(\tau_M, \tau_M; S_{\tau_M}(\tau_M)\right) \wedge T\right\}$$

can be satisfied, she immediately exits the stock markets. Based on this criterion, we derive the manager's optimal asset allocation strategy with considering a potential early stop as presented in the theorem below.

Theorem 4. Assume that $\int_0^{\hat{\tau}_M} |\theta(s)|^2 ds > 0$ and the Lagrange multipliers $(\lambda_{\hat{\tau}_M}, \gamma_{\hat{\tau}_M})$, are given by (3.6). Then there is an approximate efficient portfolio for (2.6) corresponding to early exercise time $\hat{\tau}_M$. Moreover, the efficient portfolio and associated wealth process are given respectively by:

$$\pi^*_{\hat{\tau}_M}(t) = \begin{cases} \left(\sigma(t)\sigma(t)'\right)^{-1}B(t)N\left(d_1^{\hat{\tau}_M}\left(t, S_{\hat{\tau}_M}(t)\right),\right. & \text{if } 0 \leq t \leq \hat{\tau}_M, \\ 0, & \text{if } \hat{\tau}_M < t \leq T \end{cases}$$

and:

$$X^*_{\hat{\tau}_M}(t) = \begin{cases} f_{\hat{\tau}_M}\left(t, \hat{\tau}_M; S_{\hat{\tau}_M}(t)\right), & \text{if } 0 \leq t \leq \hat{\tau}_M, \\ e^{\int_{\hat{\tau}_M}^t r(s)ds} f_{\hat{\tau}_M}\left(\hat{\tau}_M, \hat{\tau}_M, S_{\hat{\tau}_M}(\hat{\tau}_M)\right), & \text{if } \hat{\tau}_M < t \leq T. \end{cases}$$

Since the manager stops early to pursue a different goal from the investor's risk-minimization purpose, she takes higher risk before her optimal stopping time $\hat{\tau}_M$ to entrench her reputation in the labor market. This issue will be further discussed in section 4.2 by comparing the variances in different cases numerically without loss of generality.

4. Nature of the investor-manager agency conflict regarding their risk-taking behaviors

To highlight the investor-manager agency conflict in their risk-taking behaviors in fund management by illustrating the influence of optimal stopping on asset allocation strategies in the described continuous-time mean-variance framework, we follow the above two sections and keep all coefficients constant, i.e. $r(t) = r$, $\mu(t) = \mu$, $\sigma(t) = \sigma$ and $\rho(t) = \rho$, for analytical simplicity. In this section, we first present the variances of terminal wealth (and those of terminal fund assets, respectively) under different situations in the presence and absence of optimal stopping, and then compare them with each other numerically to illustrate the nature of the optimal stopping decision of the investor (and that of the manager, respectively). While doing so for both the investor and the manager, one will see that due to their different optimal stopping criteria, there exists an investor-agency conflict regarding their risk-taking behaviors.

4.1. Influence of optimal stopping on investor's risk-taking behaviors

4.1.1. Variances of terminal wealth. The variances in the investor's risk minimization problem are measured by:

$$\text{Var}[X_I(T)] = E\left[X_I(T)^2|\mathcal{F}_0| - E[X_I(T)]\mathcal{F}_0\right]^2, \tag{4.1}$$

and

$$\text{Var}\left[X_I^*(T)\right] = E\left[X_I^*(T)^2|\mathcal{F}_0| - E\left[X_I^*(T)\right]\mathcal{F}_0\right]^2, \tag{4.2}$$

in the presence of optimal stopping in theorem 3, and their analytical expressions are presented in theorem 5 below. See Appendix 3 for the detailed proof.

Theorem 5. Assume that $\int_0^T |\theta(s)|^2 ds > 0$ and that the Lagrange multipliers λ_I, γ_I are given by (3.2). Then the variance of the optimal wealth XI(T) without considering early stop is:

$$\mathrm{Var}[X_I(T)] = \gamma_I^2 e^{\int_0^T -2r(s)ds}\left(e^{\int_0^T |\theta(s)|^2)ds} - 1\right). \tag{4.3}$$

Also, the variance of the optimal wealth $X_I^*(T) = X_I^*(\tau_I)e^{\int_{\tau_I}^T r(s)ds}$ according to the investor's optimal stopping time τ_I is:

$$\mathrm{Var}\left[X_I^*(\tau_I)e^{\int_{\tau_I}^T r(s)ds}\right] = \gamma_I^2 e^{\int_0^T -2r(s)ds + \int_{\tau_I}^T 2|\theta(s)|^2 ds}\left(e^{\int_0^{\tau_I} |\theta(s)|^2 ds} - 1\right). \tag{4.4}$$

4.1.2. Nature of the optimal stopping according to the investor's criterion. As shown above, the existence of an optimal stopping time before the maturity T is influential in determining an investor's optimal asset allocation strategy. According to the essence of dynamic mean-variance framework, the expected return at the terminal time is given and the investor attempts to minimize her risk (Zhou and Li, 2000; Li *et al.*, 2001; Li and Zhou, 2006). Considering the nature of an optimal stopping as that of an American-style security, the investor is better off as she takes lower risk, given an expected terminal return. This is mainly due to the risk-return trade-off in finance markets, and it is illustrated by the following numerical example. Using this example, we show that with an optimal early stop, the risk measured by the variance of terminal wealth is lower than that in the absence of optimal stopping.

Assume that an investor who has initial wealth $X_0 = \$1$ million, invests in an investment horizon of one year (i.e. $T = 1$). Her expected terminal return is 20 percent, which will give her an expected terminal wealth $z = \$1.2$ million. To simplify the example without loss of generality, we consider an economy with one risk-free asset, which is a treasury bill with an interest rate of 6 percent, and one risky asset, which is a stock with a drift 12 percent and a volatility of 25 percent. Figure 1 presents a random path of the stock price under which Figures 2 and 3 are plotted, and Figure 2 presents the optimal wealth if early stop is (and is not, respectively) allowed. After computing the variances of terminal wealth in the absence (and presence, respectively) of optimal stopping, we plot them in Figure 3 and find that as long as the optimal stopping occurs before the terminal T, the investor is better off by taking lower risk but keeping her

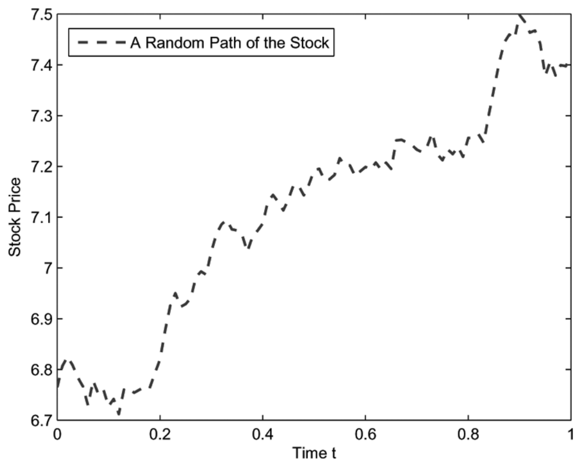

Figure 1.

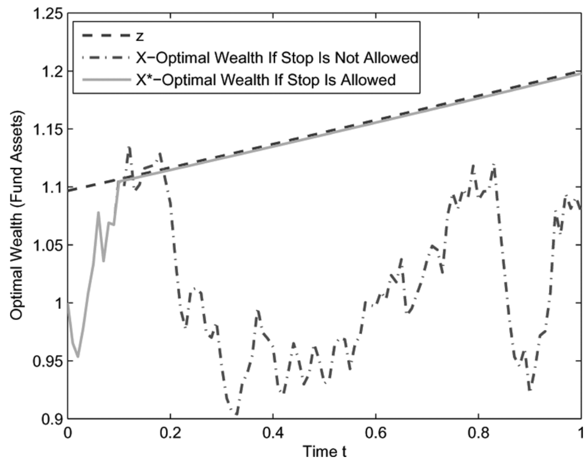

Figure 2.

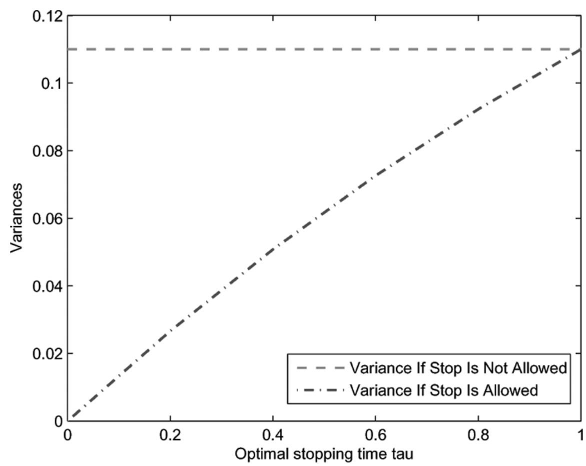

Figure 3.

expected terminal return. Furthermore, as the early stop is delayed, the gap between two variances gets smaller.

4.2. Influence of optimal stopping on the manager's risk-taking behaviors

4.2.1. Variances of terminal fund assets. The variance of the terminal fund assets in the manager's optimization problem in which early stop is allowed is measured by:

$$\begin{aligned} Var\left[X^{*}_{\hat{\tau}_M}(T)\right] &= E\left[X^{*}_{\hat{\tau}_M}(\hat{\tau}_M)^2 e^{\int_{\hat{\tau}_M}^{T} 2r(s)ds} |\mathcal{F}_0| - z^2\right] \\ &= E\left[\left(\lambda_{\hat{\tau}_M} - \gamma_{\hat{\tau}_M}\phi(\hat{\tau}_M)e^{\int_{\hat{\tau}_M}^{T} -r(s)ds}\right)_+^2 |\mathcal{F}_0|\right] - z^2 \end{aligned} \quad (4.5)$$

and its analytical expression is presented in theorem 6. See Appendix 4 for the detailed proof.

Theorem 6. Assume that $\int_0^{\hat{\tau}_M}|\theta(s)|^2ds > 0$ and that the Lagrange multipliers $\lambda_{\hat{\tau}_M}$, $\gamma_{\hat{\tau}_M}$ are given by (3.6). Then the variance of the optimal terminal wealth $X^*_{\lambda_{\hat{\tau}_M}}(T) = X^*_{\lambda_{\hat{\tau}_M}}(\lambda_{\hat{\tau}_M})e^{\int_{\lambda_{\hat{\tau}_M}}^T r(s)ds}$ according to the optimal stopping time $\lambda_{\hat{\tau}_M}$ is:

$$\begin{aligned}\mathrm{Var}\left[X^2_{\hat{\tau}_M}(T)\right] =& \lambda^2_{\hat{\tau}_M}N\left(\frac{\ln\left(\lambda_{\hat{\tau}_M/\gamma_{\hat{\tau}_M}}\right)+\int_0^T r(s)ds+\int_0^{\hat{\tau}_M}1/2|\theta(s)|^2ds}{\sqrt{\int_0^{\hat{\tau}_M}|\theta(s)|^2ds}}\right)\\ &-2\lambda_{\hat{\tau}_M}\gamma_{\hat{\tau}_M}e^{\int_0^T -r(s)ds}N\\ &\left(\frac{\ln\left(\lambda_{\hat{\tau}_M/\gamma_{\hat{\tau}_M}}\right)+\int_0^T r(s)ds+\int_0^{\hat{\tau}_M}1/2|\theta(s)|^2ds}{\sqrt{\int_0^{\hat{\tau}_M}|\theta(s)|^2ds}}\right)\\ &+\gamma^2_{\hat{\tau}_M}e^{\int_0^T -2r(s)ds+\int_0^{\hat{\tau}_M}|\theta(s)|^2ds}N\\ &\left(\frac{\ln\left(\lambda_{\hat{\tau}_M/\gamma_{\hat{\tau}_M}}\right)+\int_0^T r(s)ds+\int_0^{\hat{\tau}_M}3/2|\theta(s)|^2ds}{\sqrt{\int_0^{\hat{\tau}_M}|\theta(s)|^2ds}}\right)-z^2.\end{aligned} \quad (4.6)$$

4.2.2. Nature of optimal stopping according to the manager's criteria. Different from the investor's goal of early stop, which is to minimize risk, the manager attempts to entrench her reputation as early as possible by taking higher risk. As long as the manager stops before the terminal point of the investment horizon, therefore, the risk taken by the investment is higher than that if early stop is not allowed. This is illustrated in Figure 4 using the same numerical example as that presented in section 4.1.2. Using this example and the manager's optimal stopping criterion, we show that

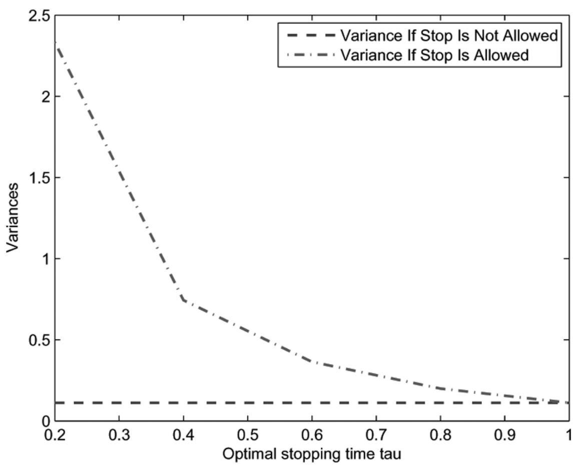

Figure 4.

with early stop, the risk measured by the variance of terminal fund assets is higher than that in the absence of optimal stopping. What is interesting is that the gap between two variances also becomes smaller when the early stop time is delayed. Note that Figure 4 is also plotted based on the random path presented in Figure 1 and the optimal fund assets under different conditions presented in Figure 2.

4.3. Discussion of the investor-manager agency conflict

The investor-manager agency conflict regarding their risk-taking behaviors has been illustrated in sections 4.1 and 4.2. Both parties may stop early, as long as their optimal stopping criteria are met. However, the investor attempts to minimize the risk, while the manager wants to entrench her reputation in the labor market as early as possible. Therefore, their optimal stopping criteria are not necessarily the same. The investor would like to stop at a certain point of time before the terminal point of the investment horizon as long as her expected terminal wealth can be guaranteed if she invests all of her wealth into risk-free assets after that. The manager has a totally different goal to pursue, and she will not stop early until the payoff of fund assets is as high as that if no early stop is allowed. Thus, the investor will take lower risk than that in the non-early stop case if she stops early, while the manager will take higher risk than that in the non-early stop case if she does so.

The above analysis first asserts that potential early stops of two parties make the investor-manager agency conflict regarding their risk-taking behaviors worsened. Second, as shown by Figures 3 and 4, one realizes that the risk increases in the length of active investment for the investor and is closer to the risk in the non-early stop case when the optimal stopping time is closer to the terminal. On the other hand, the risk decreases in the length of active investment for the manager and is also closer to the risk in the non-early stop case when the optimal stopping time is closer to the terminal. Thus, when we plot the variances of two parties' risk-taking behaviors with early stops together in Figure 5, delaying early stops of the two parties in fund management can mitigate the investor-manager agency conflict regarding their risk-taking behaviors. In other words, risk-sharing is a partial solution to this agency problem. In short, an agency conflict regarding risk-taking behaviors between two parties in fund

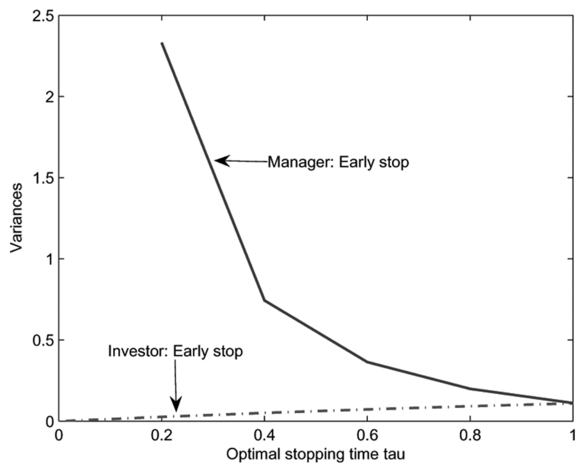

Figure 5.

management, investors and managers, is discovered in depth by the analysis above in a dynamic mean-variance economic setting, and it is an application of the investment myopia problem which is closely related to fund manager's career concerns.

5. Conclusions and recommendation for further research

Fund management becomes increasingly important in finance due to the dramatically growing number and size of fund assets. Issues in this field, such as risk measurements, fund performance, and managerial incentive-related issues have been examined by previous studies in the literature. Agency issues, however, especially the investor-manager agency conflict regarding their risk-taking behaviors, in money management has recently received attention of finance researchers in some pioneer studies (e.g. Basak, Shapiro and Tepla, 2006; Basak, Pavlova and Shapiro, 2006a, b).

This study further investigates this agency problem, which is closely related to the career concerns of fund managers and their investment myopia problems, in fund management following Lo's (2001) recommendation on measuring risk in a dynamic mean-variance framework. It shows the influences of early stop on the investor's and the manager's risk-taking behaviors, respectively, and recommends risk-sharing between two parties in fund management as a partial solution to their agency conflict. Markowitz's (1952, 1959) seminal work interprets the risk-return trade-off in a simplified way, and therefore is a landmark in the literature on financial investment and capital market theory. The portfolio selection and asset allocation theory developed in the dynamic mean-variance framework (e.g. Zhou and Li, 2000; Li *et al.*, 2001; Li and Zhou, 2006) is widely accepted by both financial economists and investors. One realizes, however, that optimal investment strategies derived from this framework assume that an investor cannot exit the stock market before the maturity, which is not realistic in financial markets. In this study, we relax this unrealistic assumption imposed by previous studies for analytical simplicity, and present optimal asset allocation strategies for both parties in fund management, the investors and the managers, considering the possibility of early stop before the terminal point of the investment horizon.

To illustrate the different influences of optimal early stopping on the investment strategy from the investor's and the manager's points of view, we compare the variances of terminal wealth under different situations about the early stop. The reason for different influences of early stop is that the investors and the managers do not necessarily have the same objectives. The investor attempts to minimize the risk, while the manager wants to entrench her reputation as early as possible. We show that, given an expected return at the end of the investment horizon, an investor can exit the stock market before the terminal point of the investment horizon and stay in the risk-free bond market only to bear a lower risk. Results show that the nature of optimal stopping provides an American-style investment problem, and with it, an investor can be better off by taking lower risk, given a fixed level of expected terminal return. However, the manager will not stop early unless her optimal stopping criterion is met, and if she stops early, the fund investment takes higher risk than that in the no-early stop case. This will make the investors worse off. Allowing early stop for both parities may either worsen the agency problem or mitigate it, depending on the time at which they stop investing in risky assets.

As one of the exploratory studies in the fund management literature on investor-manager agency conflict, several directions for future research are promising based on it. First is how the nature of optimal stopping presented in this study is applied to fund management with considering a risk-averse manager's utility maximization. Second, investment myopia problems, typical agency issues in corporate finance, have a nature of optimal early stopping, and make the principal worse off. Thus, further ways to prevent agents from exiting the stock markets early shown in this study are of importance. Third, the current study presents a one principal-one agent framework, which is less realistic than a multiple principals-multiple agents model. In fund management, if there are more than one investor and they hold different proportions of the fund assets, there is another type of agency conflict between majority and minority investors regarding the risk-taking and the free-riding issues. Thus, examining a more sophisticated model with multiple investors is also of interest. Fourth, we assume deterministic parameters in the current study for analytical simplicity, and studying how investments in individual risky assets are affected by parameters with stochastic process is interesting. Fifth, if one considers the optimal stopping examined in the current study as one of risk hedging tools, exploring its nature further in risk management is practically important.

Note

1. In the presence of possibility of optimal stopping in this problem setting, we refer it to an American-style problem. Note that, however, the optimization problem in this case is not the same as pricing American-style securities because the investor wants to minimize the variance.

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Appendix 1. Proof of theorem 1

The investor's optimization problem (2.10), according to $\tau_I = T$, is equivalent to:

$$\min E\left[X(T)^2\right] - z^2, \text{subject to} \begin{cases} E[X(T)] = z, \\ E\left[\phi(T)X(T)\right] = X_0. \end{cases} \tag{A.1}$$

As a convex optimization problem, problem (A.1) can be solved via Lagrangian method by introducing two Lagrange multipliers λ_I, $\gamma_I \in \mathbb{R}$:

$$\min E\left[X(T)^2 - 2\lambda_I X(T) + 2\gamma_I \phi(T)X(T)\right] - z^2 + 2\lambda_I z - s\gamma_I X_0, \tag{A.2}$$

where the factor 2 in front of the multipliers λ_I, γ_I is introduced in the objective function just for convenience. Clearly, this problem is equivalent to:

$$\min E\left[X(T) - \left(\lambda_I - \mu\phi(T)\right)\right]^2 - z^2 + s\lambda_I z - 2\gamma_I X_0 - E\left[\lambda_I - \gamma_I\phi(T)\right]^2, \tag{A.3}$$

in the sense that the problem (A.2) and (A.3) have exactly the same unique optimal solution:

$$x_I = \lambda_I - \gamma_I \phi(T). \tag{A.4}$$

It follows from the first equality constraint of (A.1) that only the first case of (A.4) satisfies it. Substituting this optimal solution into two equality constraints of (2.10) yields $E[\lambda_I - \gamma_I\phi(T)] = z$ and $E[\phi(T)(\lambda_I - \gamma_I\phi(T))] = X_0$, which implies the desired result (3.2).Using (2.7), one has the wealth process without early stop for the investor:

$$X_I(t) = E\left[\frac{\phi(T)}{\phi(t)} x_I | \mathcal{F}_t\right] = \lambda_I e^{\int_t^T -r(s)ds} - S_I(t), \tag{A.5}$$

where $S_I(t) = \gamma_I \phi(t) e^{\int_t^T (-2r(s) + |\theta(s)|^2)ds}$. This proves the result (3.4).

Let $X_I(t) = f(t, S_I)$, then applying Itô's formula to $f(t,S_I)$ yields:

$$df(t) = \left\{\frac{\partial f}{\partial t}(t, S_I) + \left(r(t) - \|\theta(t)\|^2\right)S_I \frac{\partial f}{\partial S_I}(t, S_I) + \frac{1}{2}|\theta(t)|^2 S_I^2 \frac{\partial^2 f}{\partial S_I^2}(t, S_I)\right\}dt. \tag{A.6}$$

Comparing with (2.11) in terms of diffusion terms yields:

$$\pi_I(t) = -\sigma(t)^{-1}\theta(t)\frac{\partial f}{\partial S_I}(t, S_I(t))S_I(t) = -\left(\sigma(t)\sigma(t)'\right)^{-1} B(t)\frac{\partial f}{\partial S_I}(t, S_I(t))S_I(t). \tag{A.7}$$

Substituting (A.7}) into (2.11) and comparing with (A.6) in terms of drift terms yield that f satisfies the following partial differential equation:

$$\begin{cases} \frac{\partial f}{\partial t}(t, S_I) + r(t)S_I \frac{\partial f}{\partial S_I}(t, S_I) + \frac{1}{2}|\theta(t)|^2 S_I^2 \frac{\partial^2 f}{\partial S_I^2}(t, S_I) = r(t)f(t, S_I), \\ f(T, S_I) = \lambda_I - S_I. \end{cases}$$

Furthermore, taking the first order for $f(t,S_I)$ in S_I yields $\partial f / \partial S_I(t, S_I) = -1$, which, together with (A.7), implies the desired results (3.3).□

Appendix 2. Proof of theorem 2

As a convex optimization problem, problem (2.10) can be solved via Lagrangian method by introducing two Lagrange multipliers $\lambda_{\tau_M},\ \gamma_{\tau_M} \in \mathbb{R}$,

$$\begin{cases} \min E\left[X(\tau_M)^2 e^{\int_{\tau_M}^{T} 2r(s)ds}\right] - z^2 - 2\lambda_{\tau_M}\left(E\left[X(\tau_M)e^{\int_{\tau_M}^{T} r(s)ds}\right] - z\right) \\ \text{subject to } X(\tau_M) \geq 0, \end{cases}$$

$$+ 2\gamma_{\tau_M}\left(E\left[\phi(\tau_M)X(\tau_M)\right] - X_0\right), \quad \text{(B.1)}$$

which is equivalent to:

$$\begin{cases} \min E\left[X(\tau_M)^2 e^{\int_{\tau_M}^{T} 2r(s)ds} - \left(\lambda_{\tau_M} - \gamma_{\tau_M}\phi(\tau_M)e^{\int_{\tau_M}^{T} -r(s)ds}\right)\right)^2\right] \\ -z^2 + 2\lambda_{\tau_M} z - +2\gamma_{\tau_M} X_0 - E\left[\left(\lambda_{\tau_M} - \gamma_{\tau_M}\phi(\tau_M)e^{\int_{\tau_M}^{T} -r(s)ds}\right)^2\right], \\ \text{subject to } X(\tau_M) \geq 0, \end{cases}$$

in the sense that Problems (B.1) and (B.2) have the same unique optimal solution:

$$x_{\tau_M}(\tau_M) = e^{\int_{\tau_M}^{T} -r(s)ds}\left(\lambda_{\tau_M} - \gamma_{\tau_M}\phi(\tau_M)e^{\int_{\tau_M}^{T} -r(s)ds}\right) + . \quad \text{(B.3)}$$

It follows from the first equality constraint of (2.12) that only the first case of (2.12) satisfies it. Substituting this optimal solution into two equality constraints of (2.12) yields:

$$p\left(\phi(\tau_M)\right) = \frac{1}{\phi(\tau_M)\sqrt{2\pi\int_t^{\tau_M}|\theta(s)|^2 ds}} \exp\left(-\frac{\left(\ln\phi(\tau_M) - \ln\phi(t)\int_t^{\tau_M}\left(r(s) + 1/2|\theta(s)|^2\right)ds\right)^2}{2\int_t^{\tau_M}|\theta(s)|^2 ds}\right). \quad \text{(B.4)}$$

According to (2.8), the density function of process $\phi(\cdot)$ is:

$$E\left[\left(\lambda_{\tau_M} - \gamma_{\tau_M}\phi(\tau_M)e^{\int_{\tau_M}^{T} -r(s)ds}\right) + |\mathcal{F}_t|\right.$$
$$\left. = \lambda_{\tau_M}\int_0^y p\left(\phi(\tau_M)\right)d\phi(\tau_M) - \gamma_{\tau_M}\int_0^y \phi(\tau_M)e^{\int_{\tau_M}^{T} -r(s)ds} p\left(\phi(\tau_M)\right)d\phi(\tau_M)\right]$$

Using (2.9), we express the first moment x_{τ_M} with the standard normal distribution as:

$$E\left[\left(\lambda_{\tau_M} - \gamma_{\tau_M\phi}(\tau_M)e^{\int_{\tau_M}^{T} -r(sy)ds}\right) + |\mathcal{F}_t|\right.$$
$$= \lambda_{\tau_M}\int_0^y p\left(\phi(\tau_M)\right)d\phi(\tau_M) - \gamma_{\tau_M}\int_0^y \phi(\tau_M)e^{\int_{\tau_M}^{T} -r(sy)ds} p\left(\phi(\tau_M)\right)d\phi(\tau_M),$$

where:

$$y = \frac{\lambda_{\tau M}}{\gamma_{\tau M} e^{\int_{\tau_M}^{T} -r(s)ds}}.$$

Furthermore:

$$\begin{aligned}
&E\left[\left(\lambda_{\tau_M} - \gamma_{\tau_M}\phi(\tau_M)e^{\int_{\tau_M}^{T} -r(s)ds}\right) + |\mathcal{F}_t|\right] \\
&= \lambda_{\tau_M}\left[1 - N\left(\frac{\ln\phi(t) - \ln\left(\lambda_{\tau_M}/\gamma_{\tau_M}\right) - \int_t^T r(s)ds - \int_t^{\tau_M} 1/2|\theta(s)|^2 ds}{\sqrt{\int_t^{\tau_M}|\theta(s)|^2 ds}}\right)\right] \\
&- \lambda_{\tau_M}\phi(t)e^{\int_t^T -r(s)ds}\left[1 - N\left(\frac{\ln\phi(t) - \ln\left(\lambda_{\tau_M}/\gamma_{\tau_M}\right) - \int_t^T r(s)ds + \int_t^{\tau_M} 1/2|\theta(s)|^2 ds}{\sqrt{\int_t^{\tau_M}|\theta(s)|^2 ds}}\right)\right] \qquad \text{(B.5)} \\
&= \lambda_{\tau_M}\left(\frac{\ln\left(\lambda_{\tau_M}/\gamma_{\tau_M}\right) - \ln\phi(t) + \int_t^T r(s)ds + \int_t^{T} 1/2|\theta(s)|^2 ds}{\sqrt{\int_t^{\tau_M}|\theta(s)|^2 ds}}\right) \\
&- \lambda_{\tau_M}\phi(t)e^{\int_t^T -r(s)ds}N\left(\frac{\ln\left(\lambda_{\tau_M}/\gamma_{\tau_M}\right) - \ln\phi(t) + \int_t^T r(s)ds + \int_t^{\tau_M} 1/2|\theta(s)|^2 ds}{\sqrt{\int_t^{\tau_M}|\theta(s)|^2 ds}}\right).
\end{aligned}$$

Using $\phi(0) = 1$, we have:

$$\begin{aligned}
&E\left[\left(\lambda_{\tau_M - \gamma_{\tau_M}}\phi(\tau_M)e^{\int_{\tau_M}^{T} -r(s)ds}\right) + |\mathcal{F}_0|\right] = \lambda_{\tau_M}N\left(\frac{\ln\left(\lambda_{\tau_M}\ \gamma_{\tau_M}\right) + \int_0^{\tau_M} 1/2|\theta(s)|^2 ds}{\sqrt{\int_0^{\tau_M}|\theta(s)|^2 ds}}\right) \\
&- \gamma_{\tau_M}e^{\int_0^T -r(s)ds}N\left(\frac{\ln\left(\lambda_{\tau_M}\ \gamma_{\tau_M}\right) + \int_0^{T} 1/2|\theta(s)|^2 ds}{\sqrt{\int_0^{\tau_M}|\theta(s)|^2 ds}}\right) = z. \qquad \text{(B.6)}
\end{aligned}$$

Similarly:

$$\begin{aligned}
&E\left[\phi(\tau_M)\right]e^{\int_{\tau_M}^{T} -r(s)ds}\left(\lambda_{\tau_M} - \gamma_{\tau_M}\phi(\tau_M)e^{\int_{\tau_M}^{T} -r(s)ds}\right) + |\mathcal{F}_0| \\
&= \lambda_{\tau_M}e^{\int_0^T -r(s)ds}N\left(\frac{\ln\left(\lambda_{\tau_M}\ \gamma_{\tau_M}\right) + \int_0^{\tau_M} 1/2|\theta(s)|^2 ds}{\int_{\tau_M}^{T}|\theta(s)|^2 ds}\right) \\
&\quad - \gamma_{\tau_M}\phi(t0)e^{\int_0^T -2r(s)ds + \int_0^{\tau_M}|\theta(s)|^2 ds}N\left(\frac{\ln\left(\lambda_{\tau_M}\ \gamma_{\tau_M}\right) + \int_0^T r(s)ds - \int_0^{T} 3/2|\theta(s)|^2 ds}{\int_{\tau_M}^{T}|\theta(s)|^2 ds}\right) \\
&= X_{0,}
\end{aligned}$$

which, together with (B.6), implies the desired result (3.6).

Using (2.7) and (2.8), one has the wealth process X_{τ_M} for $0 \le t \le \tau_M$ according to the stopping time τ_M:

$$
\begin{aligned}
X_{\tau_M}(t) &= E\left[\frac{\phi(\tau_M)}{\phi(t)}\, x_{\tau_M}(\tau_M)|\mathcal{F}_t\right] \\
&= \frac{1}{\phi(t)}\int_0^y \left(\lambda_{\tau_M}\phi(\tau_M)e^{\int_{\tau_M}^T -r(s)ds} - \gamma_{\tau_M}\phi(\tau_M)^2 e^{\int_{\tau_M}^T -2r(s)ds}\right)p\big(\phi(\tau_M)d\phi(\tau_M)\big),
\end{aligned} \tag{B.7}
$$

$$
X_{\tau_M}(t) = \lambda_{\tau_M} e^{\int_t^T -r(s)ds} N\big(-d_2^{\tau_M}(t, S_{\tau_M}(t))\big) - S_{\tau_M}(t)N\big(-d_1^{\tau_M}(t, S_{\tau_M}(t))\big), \tag{B.8}
$$

where $S_{\tau_M}(t)$, $d_1^{\tau_M}(t, S_{\tau_M}(t))$ and $d_2^{\tau_M}(t, S_{\tau_M}(t))$ are defined by (3.9). this proves the result (3.8).Let $X_{\tau_M}(t) = f(t, S_{\tau_M})$, then applying Itô's formula to $f(t, S_{\tau_M})$ yields:

$$
\begin{aligned}
df(t) =& \left\{\frac{\partial f}{\partial t}(t, S_{\tau_M}) + \left(r(t) - \|\theta(t)\|^2 S_{\tau_M}\frac{\partial f}{\partial S_{\tau_M}}(t, S_{\tau_M}) + \frac{1}{2}|\theta(t)|^2 S_{\tau_M}^2(t, S_{\tau_M})\right)\right\}dt \\
&- \theta(t')S_{\tau_M}\frac{\partial f}{\partial S_{\tau_M}}(t, S_{\tau_M})dW(t).
\end{aligned} \tag{B.9}
$$

Comparing with (2.13) in terms of diffusion terms yields:

$$
\begin{aligned}
\hat{\pi}(t) &= -\sigma(t)^{-1}\theta(t)\frac{\partial f}{\partial S_{\tau_M}}(t, S_{\tau_M}(t))S_{\tau_M}(t) \\
&= -\big(\sigma(t)\sigma(t)'\big)^{-1}B(t)\frac{\partial f}{\partial S_{\tau_M}}(t, S_{\tau_M}(t))S_{\tau_M}(t).
\end{aligned} \tag{B.10}
$$

Substituting (B.10) into (2.13) and comparing with (B.9) in terms of drift terms yield that f satisfies the following partial differential equation:

$$
\begin{cases}
\frac{\partial f}{\partial t}(t, S_{\tau_M}) = r(t)S_{\tau_M}\frac{\partial f}{\partial S_{\tau_M}}(t, S_{\tau_M}) = \frac{1}{2}|\theta(t)|^2 S_{\tau_M}^2\frac{\partial^2 f}{\partial S_{\tau_M}^2}(t, S_{\tau_M}) = t(t)f(t, S_{\tau_M}), \\
f(\tau_M, S_{\tau_M}) = (\lambda = S_{\tau_M})_+.
\end{cases}
$$

Furthermore, taking the first order for $f(t, S_{\tau_M})$) in S_{τ_M} yields $\partial f/\partial S_{\tau_M}(t, S_{\tau_M}) = -N(-d_i(t, S_{\tau_M}))$, which, together with (B.10), implies the desired results (3.7).□

Appendix 3. Proof of theorem 5

Using (2.9), we express the second moment of (4.1) without the optimal stopping in term of the standard normal distribution as:

$$
E\left[X_I(T)^2|\mathcal{F}_0\right] = \lambda_I^2 - 2\lambda_I\gamma_I e^{\int_0^T -r(s)ds} + \gamma_I^2 e^{\int_0^T -2r(s)ds + |\theta(s)|^2 ds}. \tag{C.1}
$$

The first moment is $E\left[X_I(T)|\mathcal{F}_0\right] = \lambda_I - \gamma_I e^{\int_0^T -r(s)ds}$, which, together with (C.1) implies the desired result (4.3).

Using (2.9), we express the second moment of (4.2) with the optimal stopping time τ_I

$$
\begin{aligned}
E\left[X_I^*(\tau_I)^2 e^{\int_{\tau_I}^T 2r(s)ds}|\mathcal{F}_0\right] =& \lambda_I^2 - 2\lambda_I\gamma_I e^{\int_0^T -r(s)ds\int_{\tau_I}^T |\theta(s)|^2 ds} \\
&+ \gamma_I^2 e^{\int_0^T -2r(s)ds + \int_0^T |\theta(s)|^2 ds\int_{\tau_I}^T |\theta(s)|^2 ds}
\end{aligned} \tag{C.2}
$$

Similar to (C.1) and (C.2), we have the first moment of (4.2) with the optimal stopping time τ_I: $E[X_I^*(\tau_I)e^{\int_{\tau_I}^{T} r(s)ds}|\mathcal{F}_0| = \lambda_I - \gamma_I e^{\int_0^T -r(s)ds + \int_{\tau_I}^{T} |\theta(s)|^2 ds}$, which, together with (C.2), implies the desired result (4.4).□

Appendix 4. Proof of theorem 6

Using (2.9), we express the second moment of (4.5) in term of the standard normal distribution as:

$$E\left[X^*_{\hat{\tau}_M}(\hat{\tau}_M)|\mathcal{F}_t\right] = E\left[\left(\max\left(\lambda_{\hat{\tau}_M} - \gamma_{\hat{\tau}_M}\phi(\hat{\tau}_M)e^{\int_{\hat{\tau}_M}^{T} -r(s)ds}, 0\right)\right)^2 |\mathcal{F}_t\right]$$

$$= \lambda^2_{\hat{\tau}_M}\int_0^y p\left(\phi(\hat{\tau}_M)\right)d\phi(\hat{\tau}_M) - 2\lambda_{\hat{\tau}_M}\gamma_{\hat{\tau}_M}\int_0^y \phi(\hat{\tau}_M)e^{\int_{\hat{\tau}_M}^{T} -r(s)ds} p\left(\phi(\hat{\tau}_M)\right)d\phi(\hat{\tau}_M)$$

$$+ \gamma^2_{\hat{\tau}_M}\int_0^y \phi(\hat{\tau}_M)^2 e^{\int_{\hat{\tau}_M}^{T} -2r(s)ds} p\left(\phi(\hat{\tau}_M)\right)d\phi(\hat{\tau}_M)$$

Furthermore:

$$E\left[X^*_{\hat{\tau}_M}(\hat{\tau}_M)^2|\mathcal{F}_t\right] = \lambda^2_{\hat{\tau}_M}\left(\frac{\ln\left(\lambda_{\hat{\tau}_M}\ \ \gamma_{\hat{\tau}_M}\right) - \ln\phi(t) + \int_t^{\hat{\tau}_M} 1/2|\theta(s)|^2 ds}{\sqrt{\int_t^{\hat{\tau}_M}|\theta(s)|^2 ds}}\right)$$

$$- 2\lambda_{\hat{\tau}_M}\gamma_{\hat{\tau}_M}\phi(t)e^{\int_t^T -r(s)ds} N\left(\frac{\ln\left(\lambda_{\hat{\tau}_M}\ \ \gamma_{\hat{\tau}_M}\right) - \ln\phi(t) + \int_t^T r(s)ds - \int_t^{\hat{\tau}_M} 1/2|\theta(s)|^2 ds}{\sqrt{\int_t^{\hat{\tau}_M}|\theta(s)|^2 ds}}\right)$$

$$+ \gamma^2_{\hat{\tau}_M}\phi(t)^2 e^{\int_t^{\hat{\tau}_M}|\theta(s)|^2 ds} N\left(\frac{\ln\left(\lambda_{\hat{\tau}_M}\ \ \gamma_{\hat{\tau}_M}\right) - \ln\phi(t) + \int_t^T r(s)ds - \int_t^{\hat{\tau}_M} 3/2|\theta(s)|^2 ds}{\sqrt{\int_t^{\hat{\tau}_M}|\theta(s)|^2 ds}}\right).$$

Using $\phi(0) = 1$, we have:

$$E\left[X^*_{\hat{\tau}_M}(T)^2|\mathcal{F}_0\right] = \lambda^2_{\hat{\tau}_M}\left(\frac{\ln\left(\lambda_{\hat{\tau}_M}\ \ \gamma_{\hat{\tau}_M}\right) + \int_0^T r(s)ds + \int_0^{\hat{\tau}_M} 1/2|\theta(s)|^2 ds}{\sqrt{\int_0^{\hat{\tau}_M}|\theta(s)|^2 ds}}\right)$$

$$- 2\lambda_{\hat{\tau}_M}\gamma_{\hat{\tau}_M} e^{\int_0^T -r(s)ds} N\left(\frac{\ln\left(\lambda_{\hat{\tau}_M}\ \ \gamma_{\hat{\tau}_M}\right) + \int_0^T r(s)ds + \int_0^{\hat{\tau}_M} 1/2|\theta(s)|^2 ds}{\sqrt{\int_0^{\hat{\tau}_M}|\theta(s)|^2 ds}}\right)$$

$$+ \gamma^2_{\hat{\tau}_M} e^{\int_0^T -2r(s)ds + \int_0^{\hat{\tau}_M}|\theta(s)|^2 ds} N\left(\frac{\ln\left(\lambda_{\hat{\tau}_M}\ \ \gamma_{\hat{\tau}_M}\right) + \int_0^T r(s)ds + \int_0^{\hat{\tau}_M} 3/2|\theta(s)|^2 ds}{\sqrt{\int_0^{\hat{\tau}_M}|\theta(s)|^2 ds}}\right)$$

which implies the desired result (4.6).□

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